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Two-Loop Analysis of Non-abelian Chern-Simons Theory

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Abstract

Perturbative renormalization of a non-Abelian Chern-Simons gauge theory is examined. It is demonstrated by explicit calculation that, in the pure Chern-Simons theory, the beta-function for the coefficient of the Chern-Simons term vanishes to three loop order. Both dimensional regularization and regularization by introducing a conventional Yang-Mills component in the action are used. It is shown that dimensional regularization is not gauge invariant at two loops. A variant of this procedure, similar to regularization by dimensional reduction used in supersymmetric field theories is shown to obey the Slavnov-Taylor identity to two loops and gives no renormalization of the Chern-Simons term. Regularization with Yang-Mills term yields a finite integer-valued renormalization of the coefficient of the Chern-Simons term at one loop, and we conjecture no renormalization at higher order. We also examine the renormalization of Chern-Simons theory coupled to matter. We show that in the non-abelian case the Chern-Simons gauge field as well as the matter fields require infinite renormalization at two loops and therefore obtain nontrivial anomalous dimensions. We show that the beta function for the gauge coupling constant is zero to two-loop order, consistent with the topological quantization condition for this constant.

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1 Introduction

There has recently been much interest in topological quantum field theories. They are conjectured to have something to do with a high temperature phase of quantum gravity (or superstring theory). Also of particular mathematical interest are certain models defined in three spacetime dimensions, whose actions consist purely of either (non-abelian) gauge theory or gravitational Chern-Simons terms. These have been shown to have an intimate connection with the classification theory of knots on three dimensional spaces [1] and with integrable statistical mechanics models and rational conformal field theories in two dimensions[2].

Formally, Chern-Simons theory is a strictly renormalizable quantum field theory and its perturbation expansion contains logarithmic divergences. However, the topological nature of the theory allows only trivial, finite renormalization of the Chern-Simons term itself. It is interesting to verify whether this indeed happens in the context of renormalized perturbation theory.

There is a formal proof that the correlation functions of Wilson loop operators in Chern-Simons gauge theory are topological invariants [1]. Of particular interest is the possibility that the perturbative expansion of these correlation functions defines, order by order, new topological invariants of 3-manifolds and also knot and link invariants[3]. A necessary condition for this is that the theory is at least perturbatively finite. It is a further requirement that it is invariant under gauge transformations and diffeomorphisms.

In this paper we shall examine the perturbative structure of Chern-Simons theory. We shall show that a pure Chern-Simons theory is finite to two loops and that the beta function for the gauge coupling constant vanishes to at least three loops. A novel feature of our approach is the use of regularization by dimensional reduction.

This regularization scheme renders the perturbation theory particularly simple and gauge invariant to at least three loop order. A brief version of our result appeared in ref.[4][5].

We shall also consider the renormalization of a theory where fermions and scalar matter fields couple to a Chern-Simons gauge theory. Since the matter field actions necessarily depend on the spacetime metric, this is no longer a topological field theory. Infinite renormalization is necessary and the field operators acquire nontrivial anomalous dimensions. However, we shall show that the beta function for the gauge coupling constant vanishes to two loop order [6]. This result is consistent with the topological quantization condition for the gauge coupling constant. When the gauge coupling is the only interaction and the matter is massless the matter-coupled Chern-Simons theory can be regarded as a 3 dimensional conformal field theory.

We shall devote the first part of this paper to the renormalization of pure Chern-Simons theory where the Euclidean action consists of the three-form

$$I_{C.S.} = -\frac{i\kappa}{4\pi}\int tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1.1}$$

Here the antihermitean generators of the Lie algebra are normalized so that

$$tr(T^a T^b) = -\frac{1}{2}\delta^{ab} \tag{1.2}$$

and the gauge field is a Lie algebra-valued one-form

$$A = A^a T^a \tag{1.3}$$

Under the gauge transformation

$$A \to g^{-1}(A + d)g \tag{1.4}$$

the integrand in (1.1) transforms like a closed form

$$\delta I_{C.S.} = -\frac{i\kappa}{4\pi}\int \left(d(g^{-1}dg \wedge A) + \frac{1}{3} g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \right) \tag{1.5}$$

The first term in the transformation is globally exact if A obeys suitable boundary conditions and the last term is an integer representing a winding number of the mapping, $g(x)$, of the spacetime manifold onto the gauge group [7]. For any semisimple Lie group

$$\Pi_0(\mathcal{G}) = \Pi_3(G) = Z \tag{1.6}$$

where $\mathcal{G}$ denotes the local gauge group based on the Lie group G. The gauge transform of the action is therefore a constant proportional to this integer

$$\delta I_{C.S.} = 2\pi i\kappa \times integer \tag{1.7}$$

To quantize this theory there are several issues which must be addressed. The first is gauge invariance. We consider the path integral representation of the Euclidean partition function

$$Z = \int [dA] exp(-I_{C.S.}[A]) \tag{1.8}$$

The integrand of this formal expression is only gauge invariant under large gauge transformations (1.7) if the constant κ is an integer. This is the well-known quantization condition of the topological mass term [8][7] of a topologically massive nonabelian gauge theory in three dimensions[7]. In order to do perturbative computations, it is also necessary to fix a gauge.

A second issue is ultraviolet regularization [7][9][4]. First of all, unlike a conventional field theory, the integrand in (1.8) is an oscillating function of A even in Euclidean space. Thus, even there we do not get a well-defined path integral with a convex Gaussian measure. Also, (1.1) describes a nonlinear field theory but contains no dimensional parameters. Its perturbative expansion is therefore strictly renormalizable and it is necessary to define perturbative calculations with a cutoff. It is also necessary to examine the question of renormalization of the parameters, terms in the effective action, etc.

Perturbatively (1.1) is the simplest gauge theory we know of. In the Landau gauge (see below), aside from the usual Faddeev-Popov ghosts it has the antisymmetric gloun propagator

$$\frac{4\pi}{\kappa}\delta^{ab}\frac{\epsilon_{\mu\nu\lambda}p^{\lambda}}{p^2} \tag{1.9}$$

and an antisymmetric 3-gloun vertex

$$i\frac{\kappa}{4\pi}f^{abc}\epsilon_{\mu\nu\lambda} \tag{1.10}$$

These are much more elementary in form than their counterparts in QCD for example.

In fact in an axial gauge, $n_\mu A_\mu = 0$, the theory is trivial – the ghosts decouple and the interaction term is zero. The gauge constraint is nonlinear but this is not seen perturbatively. The action is (if $n_\mu = \delta_{\mu 0}$)

$$I_{C.S.} = \frac{i\kappa}{4\pi}\int \epsilon^{0ij}A_i^a\dot{A}_j^a \tag{1.11}$$

Perturbatively this gives the naive expectation that the theory is trivial – there are no interactions and the effective action is identical to the bare action. However, the apparent triviality of the theory is naive in that the transformation to an axial gauge is not a gauge transformation under which the Chern-Simons term is invariant. It cannot been done on a compact space without introducing singularities in the gauge connection A. On an open space it has nontrivial behavior at infinity and the change in the Chern-Simons term in (1.5) is not proportional to a correctly quantized integer. Therefore the theory described by (1.1) and (1.11) are not identical. However, their perturbative structure must be very similar.

Although the above statements about triviality of the Chern-Simons theory are naive, in a slightly modified form they are likely true. This is a result of a large symmetry of (1.1) – diffeomorphism invariance. The integrand in the action is a three-form. A three-form can be written down without reference to any metric – so

(1.1) is invariant under any local deformation of the metric of the spacetime manifold. If this symmetry survives at the quantum level the only terms which could possibly appear in the effective action are those which are covariant under general coordinate transformations and which also do not depend on the metric. This means that the effective action is necessarily also a local three-form containing the operators $A \wedge dA$ and $A \wedge A \wedge A$. Furthermore the relative coefficient of these two terms is fixed by the requirement of local gauge invariance – the Chern-Simons term is gauge invariant only if the two terms occur with the same relative coefficient as in (1.1).

This means that if the symmetries of the theory under both local gauge transformations and arbitrary variations of the spacetime metric survive after quantization the only quantity in (1.1) which can change under renormalization is the overall coefficient. Furthermore, if invariance under large gauge transformations (1.7) remains a symmetry it must change in such a way that κ remains an integer.

It is impossible to fix a covariant gauge in this model without introducing a metric. It is furthermore impossible to regulate ultraviolet divergences without a metric – *i.e.* using any cutoff implies use of a distance scale which only makes sense when there is a metric. Thus the precise definition of the field theory in (1.8) requires a metric. The diffeomorphism invariance of the renormalized theory would imply that the dynamics at momentum scales much smaller than the ultraviolet cutoff are independent of the metric.

Of course, it is possible that radiative corrections generate gauge field independent but metric dependent terms in the effective action, an example being the gravitational Chern-Simons term itself [1]. If these terms are local they can be cancelled by adding local counterterms which depend only on the metric.

If there were a diffeomorphism anomaly, we also could not exclude the possibility of generating finite nonlocal terms in the effective action which would depend on the metric. An example is the term

$$\int F_{\mu\nu} \frac{1}{\sqrt{D_\lambda D^\lambda}} F^{\mu\nu} \tag{1.12}$$

If there were also a gauge anomaly, more possibilities would appear.

In this paper we shall use renormalized perturbation theory to examine whether it is in fact true that the gauge and diffeomorphism symmetries survive quantization. We shall find that, as is usual in a gauge field theory, the answer to this question depends on the regularization scheme used. We shall discover that, even though this theory is power-counting strictly renormalizable, the antisymmetric tensor structure of the vertices and gluon propagator renders it super-renormalizable. That is, we shall argue that once the tensor structure is taken into account and after suitable regularization there are only a finite number of primitively divergent Feynman diagrams.

We shall examine three types of regularization. The first introduces a Yang- Mills term

$$I_{Y.M.} = \int -\frac{1}{2e^2} tr F_{\mu\nu} F^{\mu\nu} \tag{1.13}$$

to the action. The coupling constant e^2 is dimensional and acts as an ultraviolet cutoff. This makes the model power-counting super- renormalizable at the expense of complicating the Feynman rules. There are only a finite number of divergent Feynman diagrams which must be regulated by independent means. We shall discuss several ways of doing this. We shall find that with this regularization there is a finite, correctly quantized one loop correction to the overall coefficient of the Chern-Simons action,

$$\kappa \rightarrow \kappa + \frac{C_2(G)}{2}\ \mathrm{sign}(\kappa) \tag{1.14}$$

where $C_2(G)$ is the value of quadratic Casimir operator of G in the adjoint representation. This correction has been computed before [9] and in particular Witten has derived it in a nonperturbative calculation of the phase of the determinant of the covariance in the quadratic approximation to the action [1]. The dependence on the shift of κ in (1.14) on the sign of κ was not noticed in previous literature. It is in fact necessary for covariance of the theory under orientation reversing isometrics of the manifold on which it is defined. Even though the Chern-Simons term (1.1) does not depend on the spacetime metric it changes sign under a coordinate transformation which reverses the orientation of the spacetime manifold. Thus, if the manifold has an orientation-reversing isometry the partition function (1.8) does not depend on the sign of κ. The appearance of sign(κ) in (1.14) guarantees that in the effective action one can compensate the sign reversal of the Chern-Simons term by reversing the sign of κ. (relka) is a result of our explicit calculation and implies that the magnitude of κ increases.

We then consider dimensional regularization. It has the advantage that it does not complicate the Feynman rules and higher orders in perturbation theory are more accessible. We shall show that it gives a perfectly consistent gauge invariant regularization of the theory at one-loop level. However it fails to satisfy the Slavnov-Taylor identity at two loops. We attribute this to a difficulty in the dimensional continuation of the antisymmetric tensor $\epsilon_{\mu\nu\lambda}$.

Finally we consider renormalization by dimensional reduction [4]. This is similar to the dimensional continuation used to regulate supersymmetric field theories [10]. The tensor algebra is performed in 3 dimensions to obtain scalar integrands and then the dimension of the integrations are analytically continued. This procedure is shown to satisfy the Slavnov-Taylor identities and therefore preserves the symmetries of pure Chern-Simons theory to at least three loops. We conjecture that with this regularization scheme there is no renormalization to all orders in perturbation theory, *i.e.* in line with our naive axial gauge expectation the theory turns out to be perturbatively trivial in this regularization.

In many physical applications of Chern-Simons theory the Chern-Simons gauge fields couple to either scalar or fermionic matter fields. This coupling is used as a way of implementing fractional statistics [11] of the matter fields, both in the abelian case [12] and more recently the non-abelian case [13]. The picture of the quantum Hall system as an incompressible fluid whose low energy dynamics are described by an abelian [14] [15] or non-abelian [16] Chern-Simons theory, the matter fields have a finite energy gap and the low energy limit of the theory, i.e. the effective action for excitations far below the energy gap could well be a physical realization of a topological field theory. In a finite-size system the surface states would be described by a conformal field theory [17]. Abelian Chern-Simons theory is also used to describe anyon superconductivity [18].

Matter-coupled Chern-Simons theories necessarily involve a space-time metric and are therefore not diffeomorphism invariant. In general, unlike pure Chern-Simons theory, the matter-coupled theories are not exactly solvable and it is necessary to resort to perturbative and semiclassical approximations to explore their structure. However, in certain cases, such as the models with massless matter minimally coupling to Chern-Simons fields, in which there are no dimensional parameters, the classical actions have a smaller symmetry - scale and conformal invariance. An interesting question is: whether this symmetry survives after quantization? If the answer is yes, these massless theories might be at least partially solvable [19].

Experience with quantization of *renormalizable* field theories indicates that it is likely that both the fields and the gauge coupling constants, obtain *infinite* renormalizations. Conjectures have been made in the literature about a possible renormalization group flow [20] of the statistics parameter (which contains the gauge coupling constant) in a matter-coupled abelian Chern-Simons theory. However, it is by now well established that in *abelian* Chern-Simons theory, when the matter has a mass gap the statistics parameter obtains only finite renormalization at one loop [21][22][23] and has no renormalization at all from higher loops [24][23]. When the matter is massless the Chern-Simons term has finite renormalization from two loops and beyond [23]. This has been demonstrated by explicit calculations for various kinds of matter fields[23][25][26] [27]. Higher order corrections are expected. Finiteness of the renormalization of the abelian Chern-Simons term implies that both the anomalous dimension of the abelian gauge field and the β-function for the gauge coupling must vanish. On the other hand, as we shall see in this paper, matter fields coupled to abelian Chern-Simons theory need infinite wave function renormalizations and therefore acquire non-trivial anomalous dimensions at two loops. For the abelian theory, similar conclusions have recently been reached by [28].

In this paper we shall present details of a systematic two-loop investigation of the massless and massive matter-coupled *non-abelian* Chern-Simons theories. We shall find that, in contrast to the abelian case, both the matter and the *non-abelian* gauge fields need infinite renormalization and obtain non-zero anomalous dimensions at two loops. However, the 2-loop β-functions for the gauge couplings are shown to vanish due to a delicate cancellation between the renormalization constants. As a direct consequence, the coupling constants do not run and these theories are scale independent. Since the calculations and the results involve only the divergent renormalization parts, they are independent of regularization schemes, and are valid for the massive matter-coupled Chern-Simons theories as well. Furthermore, for the massless theories, the classical Callan-Symanzik equations are modified merely by the anomalous dimensions, and then the conformal and scale invariance survives quantization to two-loop order.

In Section 2 we discuss the general structure of the renormalization of pure Chern-

Simons theory, establish the notation and discuss Slavnov-Taylor identities. In Section 3 we examine its one-loop structure. In Section 4 we give details of two-loop renormalization of the pure Chern-Simons theory. In Section 5 we include the effects of matter fields to two-loop order. Section 6 contains a discussion of the results.

2 Pure Chern-Simons Theory: Slavnov-Taylor Identities, Power Counting, and Renormalization Constants

In order to do perturbation theory we must fix the gauge. We shall use a linear covariant gauge

$$\partial^\mu A_\mu = 0 \tag{2.1}$$

as this sort of gauge condition can be realized on a compact space. This requires introducing Faddeev-Popov ghosts through the standard procedure. The ghost action is

$$I_g = \int tr \partial^\mu \bar{c} D_\mu c - \frac{1}{\beta}(\partial^\mu A_\mu)^2 \tag{2.2}$$

with

$$D_\mu = \partial_\mu + [A, \quad] \tag{2.3}$$

$$c = c^a T^a \tag{2.4}$$

It gives rise to the Feynman rules

$$\delta^{ab}\frac{1}{p^2} \tag{2.5}$$

for the ghost propagator,

$$i f^{abc} p_\mu \tag{2.6}$$

for the ghost-ghost-gluon vertex as usual. For each ghost loop, a minus sign will be added to the Feynman integral. In this work, the Landau gauge $\beta = 0$ is used exclusively.

Now we consider the ultraviolet behavior of the theory. Since the three- and higher-point functions are ultraviolet convergent, we concentrate on the self-energies. With the propagators and vertices in (refru1), (refru2), (refru3), and (refru4), naive powercounting shows that both the gauge field and ghost self-energies are linearly divergent. However, more careful study will show that it is not the case [39].

First of all, the ghost self-energy diagrams with an odd number of loops are identically zero and the ones with an even number of loops are potentially logarithmically divergent. The reasons are the following: Any ghost self-energy diagram with an odd number of loops contains an odd number of antisymmetric ϵ-tensors, so that after index contractions, one and only one of them remains. Further, by the tensor structure, the three Lorentz indices of the remaining ϵ-tensor must be contracted with momenta. However, there is only one external momentum in the self-energy diagram so that the contraction gives zero. On the other hand, the diagrams with even numbers of loops contain an even number of ϵ-tensors. The contractions among the Lorentz indices will exhaust all ϵ-tensors so that they have normal contributions to the ghost self-energy, which takes the form of $\tilde{\Pi}(p)p^2$. Consequently, $\tilde{\Pi}(p)$ and therefore the ghost self-energy is potentially logarithmically divergent.

Secondly, the gluon self-energy has dimension one. However, the local Euclidean invariant linearly divergent tensor $\Lambda\delta_{\mu\nu}\delta^{ab}$ is not allowed by gauge symmetry. The allowed tensor structure is

$$\Pi_e(p)(\delta_{\mu\nu}p^2 - p_\mu p_\nu) + \Pi_o(p)\epsilon_{\mu\nu\lambda}p^\lambda \tag{2.7}$$

Once the projection operators $\delta_{\mu\nu}p^2 - p_\mu p_\nu$ and $\epsilon_{\mu\nu\lambda}p^\lambda$ are extracted, the Feynman integrals contributing to $\Pi_e(p)$ are finite and to $\Pi_o(p)$ are potentially logarithmically divergent [30]. Moreover, the diagrams with even numbers of loops contain odd numbers of ϵ-tensors and thus contribute only to $\Pi_o(p)$. Diagrams with odd numbers of loops contain even numbers of ϵ-tensors and contribute only to $\Pi_e(p)$. Thus we see that one loop and all odd numbers of loops are actually convergent and might give corrections only to the symmetric part of the gluon self-energy. However, we should be aware that any non-zero contribution to $\Pi_e(p)$ would reflect a diffeomorphism anomaly since it implies the quantized theory is not independent of spacetime metric any more. On the other hand, the logarithmic divergences may appear only in even numbers of loops.

As a regulator one may add a Yang-Mills term (1.13) to the Chern-Simons action (1.1). Then the bare gluon propagator (1.9) is replaced by

$$\Delta_{\mu\nu}(p) = \frac{4\pi}{\kappa}\frac{\mu}{p^2(p^2+\mu^2)}(\mu\epsilon_{\mu\nu\lambda}p^\lambda + \delta_{\mu\nu}p^2 - p_\mu p_\nu) \tag{2.8}$$

and the three-gluon vertex (1.10) by

$$i\frac{\kappa}{4\pi}\frac{1}{\mu}f^{abc}[\mu\epsilon_{\mu\nu\lambda} - (r-q)_\mu\delta_{\nu\lambda} - (q-p)_\lambda\delta_{\mu\nu} - (p-r)_\nu\delta_{\lambda\mu}] \tag{2.9}$$

where the dimensional parameter $\mu = \frac{\kappa}{4\pi}e^2$ is the topological mass of the gluon [7]) and here plays the role of a cutoff to be taken to infinity at the end of the calculation. This regularization also introduces a new interaction – the four-gluon vertex

$$\frac{\kappa}{4\pi}\frac{1}{\mu}[f^{abe}f^{cde}(\delta_{\mu\lambda}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\lambda}) + f^{cbe}f^{ade}(\delta_{\mu\lambda}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\lambda}) + f^{dbe}f^{cae}(\delta_{\mu\nu}\delta_{\lambda\sigma} - \delta_{\mu\sigma}\delta_{\nu\lambda})] \tag{2.10}$$

With finite μ the theory is power counting super-renormalizable. Only a finite number of diagrams are potentially ultraviolet divergent. Moreover, the new Feynman rules have a more complicated tensor structure, therefore they give rise to more diagrams.

Since the tensor structure of the propagators and vertices is altered in this renormalization scheme, the naive power-counting arguments given above are not strictly correct. However, we still believe that they should apply to the divergent parts of the quantum corrections. It does not necessarily apply to finite parts and, as we shall see in the next Section, the coefficient of the Chern-Simons term is renormalized by a finite amount at one loop in this regularization scheme.

In general the perturbative expansions of QED_3 and QCD_3 have infrared divergences. However, three dimensional Chern-Simons gauge theories are infrared finite (at least) in the Landau gauge. The essential reason is that the gluon or photon propagator (see (1.9)) behaves like p^{-1}, instead of p^{-2}, as $p \to 0$. This property makes the propagator integrable at small momenta in three dimensions. With the F^2 regularization the theory looks like a massive gauge field theory, which has been shown to be free of infrared divergences [7][9].

In addition to F^2 regularization one may use regularization by dimensional continuation. Namely one dimensionally continues the loop-momentum integrals to non-integral dimensions to make them convergent. The advantage of working with this regularization is that one does not need to complicate the Feynman rules at all and higher orders in perturbation theory are more accessible. However, it is difficult to dimensionally continue the three-index ϵ-symbol appearing in the Chern-Simons action.

Technically, what is needed is a rule for contracting indices of the ϵ-tensors when extracting external tensor structures from a Feynman integral or performing tensor contractions to obtain scalar integrands. Our procedure is to first perform all tensor contractions to obtain scalar integrands. Then we define the singularity of ultraviolet divergent integrals using dimensional continuation.

But when contracting the tensor indices of the $\delta_{\mu\nu}$ obtained from the contraction of two ϵ-symbols, logically there are still two possible choices: Is it in the continued dimensions $\omega = 3 - \epsilon$ or in the physical dimensions $D = 3$? In the first case the contractions are done in $\omega = 3 - \epsilon$ dimensions, *e.g.*

$$\epsilon_{\mu\sigma\eta}\epsilon^{\mu\lambda\tau} = (\delta_\sigma^\lambda \delta_\eta^\tau - \delta_\sigma^\tau \delta_\eta^\lambda)\Gamma(\omega - 1) \tag{2.11}$$

$$\delta_\sigma^\sigma = \omega \tag{2.12}$$

Another possibility is performing the tenser algebra in the strict three dimensions, *i.e.*

$$\epsilon_{\mu\sigma\eta}\epsilon^{\mu\lambda\tau} = (\delta_\sigma^\lambda \delta_\eta^\tau - \delta_\sigma^\tau \delta_\eta^\lambda) \tag{2.13}$$

$$\delta_\sigma^\sigma = 3 \tag{2.14}$$

We shall call the first choice dimensional regularization and the second regularization by dimensional reduction. The two choices do not make any difference in the pole terms in $\omega - 3$, but may lead to different finite parts. A priori it is unclear which choice should be used; also it is unclear whether either of them maintains gauge invariance. We shall show below by explicit calculations that, dimensional reduction leads to gauge invariant results in pure Chern-Simons theory at least to three loops and in matter-coupled Chern-Simons theory at least to two loops, but dimensional regularization violates the Slavnov-Taylor identities satisfied by renormalization constants in pure Chern-Simons theory at two loops.

Now let us define relevant renormalization constants and establish the Slavnov-Taylor identities among them. With the F^2 regularization, the inverse gluon propagator is

$$\delta^{ab}\Delta^{-1}_{\mu\nu}(p) = \delta^{ab}[\frac{\kappa}{4\pi} Z_A(p)\epsilon_{\mu\nu\lambda}p^\lambda + Z'_A(p)(\delta_{\mu\nu}p^2 - p_\mu p_\nu)] \tag{2.15}$$

where

$$Z_A(p) = 1 + \Pi_o(p) \tag{2.16}$$

$$Z'_A(p) = \frac{1}{\mu} Z''_A(p) = \frac{1}{\mu}(1 + \Pi_e(p)) \tag{2.17}$$

(When regularization by dimensional continuation is used, $Z'_A(p)$ will be identified to $\Pi_e(p)$, which has the dimension of mass^{-1}, see (2.7)). Then the full gluon propagator is

$$\Delta_{\mu\nu}(p) = \frac{4\pi}{\kappa} \frac{\mu}{Z''_A(p)p^2(p^2 + (\frac{Z_A(p)}{Z''_A(p)})^2\mu^2)} (\frac{Z_A(p)}{Z''_A(p)} \mu \epsilon_{\mu\nu\lambda} p^\lambda + \delta_{\mu\nu} p^2 - p_\mu p_\nu) \tag{2.18}$$

Similarly, the full three-gluon vertex is

$$i\frac{\kappa}{4\pi}\Gamma^{abc}_{\mu\nu\lambda} = \frac{\kappa}{4\pi} f^{abc}(Z_g(p,q,r)\epsilon_{\mu\nu\lambda} - Z'_g(p,q,r)[(r-q)_\mu \delta_{\nu\lambda} + cyclic] + ...) \tag{2.19}$$

where

$$Z_g(p,q,r) = 1 + T_g(p,q,r) \tag{2.20}$$

$$Z'_g(p,q,r) = \frac{1}{\mu}(1 + T'_g(p,q,r)) \tag{2.21}$$

(As above, if regularization by dimensional continuation is used, Z'_g is identified with T'_g.) To identify the gluon wave function and the three-gluon vertex renormalization functions from $Z_A(p)$, Z'_A, $Z_g(p)$, and $Z'_g(p)$, we need to first remove the regularization cut-off $\mu \to \infty$. Suppose $Z''_A(p)$ and $Z'_g(p)$ are convergent in this limit. As we shall see in the subsequent Sections, it is always the case; then

$$\Delta_{\mu\nu}(p) \to \frac{4\pi}{\kappa} \frac{1}{Z_A(p)} \frac{\epsilon_{\mu\nu\lambda} p^\lambda}{p^2} \tag{2.22}$$

$$\Gamma^{abc}_{\mu\nu\lambda} \to f^{abc} Z_g(p,q,r) \epsilon_{\mu\nu\lambda} \tag{2.23}$$

It shows that the renormalization functions are $Z_A(p)$ and $Z_g(p,q,r)$. When regularization by dimensional continuation is used, Π_e and T'_g are convergent. But any non-zero contribution to them would imply the space-time-metric dependence of the quantized theory and therefore an anomaly of the diffeomorphism invariance. We shall find that Π_e and T'_g get null contributions in the dimensional reduction regularization scheme up to two loops.

The renormalized ghost propagator and the $\bar{c}Ac$ vertex are

$$\tilde{\Delta}^{ab}(p) = \frac{\delta^{ab}}{Z_{gh}(p)p^2} \tag{2.24}$$

$$Z_{gh}(p) = 1 + \tilde{\Pi}(p) \tag{2.25}$$

and

$$i\tilde{\Gamma}^{abc}_{\lambda}(p,q,r) = if^{abc}\tilde{Z}_g(p,q,r)p_\lambda \tag{2.26}$$

$$\tilde{Z}_g(p,q,r) = 1 + \tilde{\Gamma}(p,q,r) \tag{2.27}$$

An important fact we shall see in the next Sections is that these renormalization functions are in fact finite constants, *i.e.* they are independent of both cut-off and external momenta:

$$Z_A(p) = Z_A, \quad Z_g(p,q,r) = Z_g, \quad \textit{and so on} \tag{2.28}$$

This means the theory has at most a finite renormalization, which needs no choice of a renormalization point. As a result, the Slavnov-Taylor identity among these renormalization functions is simply a relation of the constants

$$\frac{Z_A}{Z_{gh}} = \frac{Z_g}{\tilde{Z}_g} \tag{2.29}$$

(2.29) reflects the invariance under "small" gauge transformations and will provide a consistency check of the regularization methods in the next Sections.

With the renormalization constants Z_A and Z_g, the Chern-Simons action has the form

$$S = -\frac{i\kappa}{4\pi}\int tr(Z_A A\wedge dA + \frac{2}{3}Z_g A\wedge A\wedge A) \tag{2.30}$$

$$= -\frac{i\kappa_r}{4\pi}\int tr(A_r\wedge dA_r + \frac{2}{3}A_r\wedge A_r\wedge A_r) \tag{2.31}$$

where we have defined the renormalized field A_r,

$$A_r = \frac{Z_g}{Z_A} A \tag{2.32}$$

and the renormalized Chern-Simons coefficient κ_r,

$$\kappa_r = \frac{Z_A^3}{Z_g^2} \kappa \tag{2.33}$$

Moreover, the invariance under large gauge transformations requires the renormalized Chern-Simons coefficient, like the bare one, to be quantized

$$\kappa_r = integer \tag{2.34}$$

We shall check in the next Sections whether the three regularization schemes we are considering preserve the quantization condition (2.34).

3 One-Loop Structure of Pure Chern-Simons Theory

A. F^2 Regularization

An important one loop result obtained with the F^2 regularization is a finite renormalization of the coefficient of the Chern-Simons action by an integer, $\kappa_r = \kappa + \frac{1}{2} C_2(G) \mathrm{sign}(\kappa)$. Thus the renormalized coefficient κ_r satisfies the topological quantization condition (2.34). This shift can also be obtained using Pauli-Villars regularization [31]. We shall find in the next subsection this shift of κ is absent when we use regularization by dimensional continuation.

F^2 regularization is more effective at making higher order diagrams convergent, as shown by naive power counting. However, the Yang-Mills term regulates only a part of, but not all, the divergences in the one loop diagrams. Therefore a complement is required at this level. We shall use regularization by dimensional continuation when necessary. We shall find that at one loop we need to use dimensional regularization for integrals which contain no ϵ-tensors.

One-loop renormalization of topologically massive gauge theories has been discussed in previous literature [7][9][32]. Here we shall review the one-loop structure of these theories as a F^2 regularization of Chern-Simons theory. The latter is recovered in the limit as the cutoff μ goes to infinity. A new aspect of our discussion is the appearance of 'sign(κ)' in the shift of κ.

Let us consider the one loop diagrams in Fig.1. Using F^2 regularization, the Feynman rules are the altered ones, (2.8), (2.9), and (2.10) with (2.5) and (2.6). First we calculate the ghost self-energy. The diagram (1.d) is

$$\tilde{\Pi}^{(1)}(p) = \frac{4\pi}{\kappa}\frac{C_2(G)}{2p^2}\mu\int\frac{d^3k}{(2\pi)^3}\frac{(k.p)^2 - k^2p^2}{k^2(k+p)^2(k^2+\mu^2)} \tag{3.1}$$

where we have used the convention for gauge group indices

$$f^{adc}f^{dcb} = \delta^{ab}\frac{C_2(G)}{2} \tag{3.2}$$

The integral in (refpie) is finite with the le ading term proportional to $1/|\mu|$. Evaluating it then letting μ go to ∞, we get

$$\tilde{\Pi}^{(1)}(p) = \tilde{\Pi}^{(1)}(0) = -\frac{2}{3\kappa}\frac{C_2(G)}{2}\text{sign}(\kappa) \tag{3.3}$$

Then we have the one loop ghost wave function renormalization constant

$$Z^{(1)}_{gh} = 1 - \frac{2}{3\kappa}\frac{c_2(G)}{2}\text{sign}(\kappa) \tag{3.4}$$

By the symmetries, the gluon self-energy $\Pi^{(1)}_{\mu\nu}(p)$ can be separated as

$$\Pi^{(1)}_{\mu\nu}(p) = \frac{1}{\mu}\Pi^{(1)}_e(p)(\delta_{\mu\nu}p^2 - p_\mu p_\nu) + \frac{\kappa}{4\pi}\Pi^{(1)}_o(p)\epsilon_{\mu\nu\lambda}p_\lambda \tag{3.5}$$

$\Pi_o^{(1)}(p)$ yields contribution from the part of the gluon loop (1.a) which has odd number of ϵ-tensors. $\Pi_e^{(1)}(p)$ obtains contributions from the ghost loop (1.b), the tadpole (1.c), and the part of (1.a) with even number of ϵ-tensors. Contracting $\Pi_{\mu\nu}^{(1)}(p)$ with $\frac{\kappa}{4\pi}\epsilon_{\mu\nu\lambda}p^\lambda/2p^2$ and $\mu\delta_{\mu\nu}/2p^2$, we obtain $\Pi_o^{(1)}(p)$ and $\Pi_e^{(1)}(p)$

$$\Pi_o^{(1)}(p) = \frac{4\pi}{\kappa}\frac{C_2(G)}{2p^2}\mu\int\frac{d^3k}{(2\pi)^3}\frac{(k^2p^2-(k.p)^2)(5k^2+5(k.p)+4p^2+2\mu^2)}{k^2(k+p)^2(k^2+\mu^2)((k+p)^2+\mu^2)} \tag{3.6}$$

$$\Pi_e^{(1)}(p) = -\frac{C_2(G)}{8p^2}\mu\int\frac{d^3k}{(2\pi)^3}[\frac{N_e(p,k)}{k^2(k+p)^2(k^2+\mu^2)((k+p)^2+\mu^2)}+\frac{2\mu}{\pi}] \tag{3.7}$$

where

$$\begin{aligned} N_e(p,k) &= 6k^6+18k^4(k.p)+20k^4p^2+22k^2(k.p)p^2-12(k.p)^3+9k^2p^4 \\ &\quad -7(k.p)^2p^2+\mu^2[2k^4+4k^2(k.p)+k^2p^2+(k.p)^2] \end{aligned} \tag{3.8}$$

The integration of (3.7) is convergent. Doing it then taking $\mu\to\infty$, we have

$$\Pi_o^{(1)}(p) = \Pi_o^{(1)}(0) = \frac{7}{3\kappa}\frac{C_2(G)}{2}\mathrm{sign}(\kappa) \tag{3.9}$$

The integration of (3.8) is power-counting linearly divergent. But the divergence belongs to the kind that violates the gauge invariance. Therefore any regularizations which preserve the gauge symmetry, such as regularization by dimensional continuation, will remove it. The calculation further indicates that when $\mu\to\infty$, (3.8) is finite. This verifies the argument in the last Section that the (one loop) gluon wave function renormalization constant is $Z_A^{(1)}$. It is given by

$$Z_A^{(1)} = 1+\Pi_o^{(1)} = 1+\frac{7}{3\kappa}\frac{C_2(G)}{2}\mathrm{sign}(\kappa) \tag{3.10}$$

Now let us proceed to show that the net result of one-loop corrections to the $\bar{c}Ac$ vertex vanishes as $\mu\to\infty$: The one-ϵ term of (1.i) cancels against the three-ϵ term of (1.h); The one-ϵ term of (1.h) goes to zero; The non-ϵ terms in the two diagrams cancel each other. This is in agreement with the general arguments that [33] to any order the $\bar{c}Ac$ vertex renormalization constant is always

$$\tilde{Z}_g^{(1)} = 1 \tag{3.11}$$

To get the one loop three-gluon vertex renormalization constant, an easy way is to exploit the Slavnov-Taylor identity (2.29). Substituting (3.4), (3.10), and (3.11) into (2.29), we obtain

$$Z_g^{(1)} = \frac{Z_A^{(1)}}{Z_{gh}^{(1)}} \tilde{Z}_g^{(1)} = 1 + \frac{3}{\kappa} \frac{C_2(G)}{2} \mathrm{sign}(\kappa) \tag{3.12}$$

where we have used the assumption that κ is large enough so that the perturbative expansion makes sense. The direct calculation of diagrams (1.e), (1.f), and (1.g) gives the same result.

Substituting the renormalization constants $Z_A^{(1)}$ and $Z_g^{(1)}$ in (2.33), we have

$$\kappa_r^{(1)} = \frac{(Z_A^{(1)})^3}{(Z_g^{(1)})^2} \kappa = \kappa + \frac{1}{2} C_2(G) \mathrm{sign}(\kappa) \tag{3.13}$$

It implies that the renormalized Chern-Simons coefficient satisfies the topological quantization condition if the bare one does. That is, quantization preserves the large gauge invariance at one loop.

B. Regularization by Dimensional Continuation

Using regularization by dimensional continuation, we have the simpler Feynman rules, (1.9), (1.10), (2.5), and (2.6). Further, the diagrams with four-point vertex in Fig. 1 do not appear. Let us do a tensor-structure analysis first. It is easy to see that the ghost self-energy diagram (1.d) is zero because of the ϵ-tensor.

The gluon self-energy (1.a) and (1.b) has zero contribution to Π_o but might contribute to Π_e. For the same reason, the three-gluon vertex (1.e) and (1.f) do not contribute to the antisymmetric part of three-gluon vertex, T_g, but could contribute to its symmetric part T'_g; On the other hand, being diagrams with odd number of ϵ-tensors, the $\bar{c}Ac$ vertex diagrams (1.h) and (1.i) might develop antisymmetric parts. All of these parts, if nonvanishing, would give rise to vertices in the effective action which violate the diffeomorphism invariance of the theory.

However, we find that the contributions to the gluon self-energy, the three-gluon vertex, and the $\bar{c}Ac$ vertex, respectively, are all canceled pairwise:

$$
\begin{aligned}
(1.b) &= -(1.a) = -\delta^{ab}\frac{C_2(G)}{2}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{(k+p)_\mu k_\nu + k_\mu (k+p)_\nu}{k^2(k+p)^2} && (3.14)\\
(1.i) &= -(1.h) = i\frac{4\pi}{\kappa}T^{abc}\int\frac{d^3k}{(2\pi)^3}\frac{k^\lambda\epsilon_{\sigma\eta\tau}r^\sigma q^\tau k^\eta}{k^2(k+q)^2(k-r)^2} && (3.15)\\
(1.f) &= -(1.e) = iT^{abc}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{(k-r)_\mu(k+q)_\nu k_\lambda + (k+q)_\mu k_\nu (k-r)_\lambda}{k^2(k+q)^2(k-r)^2} && (3.16)
\end{aligned}
$$

where $T^{abc} = f^{ead}f^{dbg}f^{gce}$ and $r+p+q=0$.

Thus, with dimensional regularization at one loop there is no renormalization to the Chern-Simons action, *i.e.*

$$Z_A^{(1)} = Z_{gh}^{(1)} = Z_g^{(1)} = \tilde{Z}_g^{(1)} = 1 \tag{3.17}$$

and no contribution to the finite part of all other vertices.

4 Two-Loop Structure of Pure Chern-Simons Theory

In this Section we shall use both the dimensional regularization and the dimensional reduction to compute the two-loop diagrams of Fig. 2. We shall find that the dimensional regularization fails to obey the Slavnov-Taylor identity and therefore breaks the gauge symmetry, while the dimensional reduction preserves the gauge invariance and gives no renormalization of the Chern-Simons action.

Contrary to the case at one loop, the two-loop gluon self-energy and three-gluon vertex diagrams are anti-symmetric under exchange of Lorentz indices. Therefore it is possible that two-loop corrections renormalize the Chern-Simons action. Also, the two-loop ghost self-energy and $\bar{c}Ac$ vertex diagrams contain even number of ϵ-tensors so that they might renormalize the gauge fixing action. Consequently, at this order what we need to find is the possible leading corrections to the renormalization constants Z_A, Z_g, Z_{gh}, and $\tilde{Z}_g$.

To start, let us argue that the three-point vertices vanish identically at this order, namely

$$Z_g^{(2)} = \tilde{Z}_g^{(2)} = 1 \tag{4.1}$$

We shall discuss separately the non-planar and the planar three-legged diagrams.

First of all, the non-planar diagrams in Fig.2 vanish individually because of the symmetry of the group indices. To explain this point, choose one such diagram and cut any one of its two crossing internal propagators. In this way, we get a one-loop four-point vertex which connects with a bare three-point vertex through a propagator. The group factor of the one-loop four-point vertex takes the form of $T^{abcd} = f^{eag} f^{gbh} f^{hci} f^{ide}$. It is easy to check that T^{abcd} is symmetric under the exchange of two indices which are not neighbors, *i.e.* $T^{abcd} = T^{adcb} = T^{cbad}$. On the other hand, a bare three-point vertex carries a group factor, the structure constant f^{abc}, which is anti-symmetric under the exchange of any two indices. By the structure of the diagrams, the total group factor for a non-planar three-legged diagram is $N^{abc} = trT^{ajcd} f^{bjd}$. The contraction in the group indices gives zero, therefore the non-planar diagram vanishes.

Secondly, all planar (three-legged) diagrams cancel in pairs. The cancellations occur as a result of the cancellations at one loop. To see this clearly, we label the planar three-legged diagrams in Fig. 2 that are canceled in pair by same letter in lower and upper cases, such as (l) and (L), (m) and (M), and so on. In each of such pairs, the two diagrams differ from each other only in a one-loop sub-diagram, and the two one-loop sub-diagrams just form a pair that cancel against each other, i.e. being one of the three pairs in (3.14), (3.15), and (3.16).

We should emphasize that the cancellations mentioned above hold without regularization ambiguity, because all two-loop three-legged diagrams are convergent. In other words, *the vanishing of the two-loop three-legged diagrams is independent of regularization*.

Now we consider the two-point functions, Fig. (2a)–(2k). First we note that Fig. (2d) and (2k) vanish identically, because each contains the one-loop ghost self-energy as a sub-diagram, which is known to vanish with dimensional regularization (see Sec. 3B). Since the rest of the two-loop self-energy diagrams, Fig. (2a-2j), can be paired in a way similar to the three-legged diagrams, it seems that the above pairwise cancellation would happen too. However we must be very careful here since the two-legged diagrams (2a) – (2j) are logarithmically divergent. In regularization by dimensional continuation, the singularity is expressed by a single pole of the Gamma function, $\Gamma(\frac{3-\omega}{2})$ with $\omega \to 3$. On the other hand, depending on the rules, the contractions of Lorentz indices might generate factors like $3-\omega$ when two diagrams are added. In this case, the cancellation can be incomplete: although the pole terms are summed to zero, a finite term may remain.

With dimensional regularization, we shall use the contraction rules (2.11) (2.12 to write down the Feynman integrations. Upon contracting with $\frac{4\pi}{\kappa}\epsilon_{\mu\nu\lambda}p^\lambda$, the diagrams

(2.a) – (2.c) are

$$
\begin{aligned}
(2.a) &= \frac{1}{2}\frac{\omega-1}{2}[\Gamma(\omega-1)]^4(\frac{4\pi}{\kappa})^2 R^{ab}\frac{1}{p^2}I_1(\omega) && (4.2)\\
(2.b) &= -\Gamma(\omega-1)(\frac{4\pi}{\kappa})^2 R^{ab}\frac{1}{p^2}I_1(\omega) && (4.3)\\
(2.c) &= \frac{1}{2}(\frac{4\pi}{\kappa})^2 R^{ab}\frac{1}{p^2}I_1(\omega) && (4.4)
\end{aligned}
$$

where the factors $\omega - 2$ and $\Gamma(\omega - 1)$ in (2.a) and (2.b) are from the contractions of ϵ-tensors, and

$$R^{ab} = R\delta^{ab} = f^{ade}f^{dcg}f^{ghe}f^{hcb} \tag{4.5}$$

and

$$
\begin{aligned}
I_1(\omega) &= \Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{F(k,p)}{k^2(k+p)^2} && (4.6)\\
F(k,p) &= \int\frac{d^3q}{(2\pi)^3}\frac{\epsilon_{\sigma\eta\lambda}p^\sigma k^\eta q^\lambda \epsilon_{\tau\theta\xi}p^\tau k^\theta q^\xi}{q^2(q+k)^2(q-p)^2} && (4.7)
\end{aligned}
$$

The integration over q is convergent, giving

$$F(k,p) = \frac{1}{32}[k^2|p| + |k|(p^2 + k.p) - |k+p|(k.p) + |p|(k.p) - |p||k||k+p|] \tag{4.8}$$

Substituted F(k,p) into $I_1(\omega)$, the integration over k gives a pole term

$$\Gamma(\frac{3-\omega}{2})\frac{p^2}{16}\frac{1}{(4\pi)^{\frac{\omega}{2}}\Gamma(\frac{1}{2})}\int_0^1 dx\frac{\sqrt{x}}{(x(1-x)p^2)^{\frac{\omega-3}{2}}} \tag{4.9}$$

On the other hand, adding together (2.a), (2.b), and (2.c), we have a factor

$$\frac{1}{2}[1 - 2\Gamma(\omega-1) + \frac{\omega-1}{2}(\Gamma(\omega-1))^4] \tag{4.10}$$

which annihilates the contribution of any finite parts of the integral $I_1(\omega)$. Then using

$$[1 - 2\Gamma(\omega-1) + \frac{\omega-1}{2}(\Gamma(\omega-1))^4]\Gamma(\frac{3-\omega}{2}) = -4(1-\gamma) - 1 \tag{4.11}$$

as $\omega \to 3$, where $\gamma = 0.5772...$ is the Euler constant, we obtain

$$-\frac{1}{384\pi^2}(5-4\gamma)(\frac{4\pi}{\kappa})^2 R \tag{4.12}$$

as the contribution to $\Pi_o^{(2)}$ of the diagrams (2.a), (2.b), and (2.c).

The gluon self-energy diagrams (2.e) and (2.f) give

$$(2.e) \;=\; -(\frac{4\pi}{\kappa})^2(\frac{C_2(G)}{2})^2\delta^{ab}\frac{1}{p^2}I_2(\omega) \tag{4.13}$$

$$(2.f) \;=\; \frac{\omega-1}{2}(\Gamma(\omega-1))^2(\frac{4\pi}{\kappa})^2(\frac{C_2(G)}{2})^2\delta^{ab}\frac{1}{p^2}I_2(\omega) \tag{4.14}$$

where $I_2(\omega)$ has a pole

$$\begin{aligned} I_2(\omega) &= (\omega-1)(\Gamma(\omega-1))^2\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{p^\sigma\epsilon_{\sigma\tau\lambda}k^\lambda(-\frac{1}{32}|k|\delta^{\tau\eta})\epsilon_{\eta\xi\theta}p^\xi k^\theta}{k^2k^2(k+p)^2} \\ &= -\frac{1}{32}(\omega-1)(\Gamma(\omega-1))^3\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{(k.p)^2-k^2p^2}{|k|^3(k+p)^2} \qquad (4.15) \\ &= \frac{1}{32}\frac{1}{8\pi^2}(\omega-1)^2(\Gamma(\omega-1))^3\Gamma(\frac{3-\omega}{2})p^2\int_0^1 dx\frac{\sqrt{x}}{(x(1-x)p^2)^{\frac{3-\omega}{2}}} \qquad (4.16) \end{aligned}$$

(2.e) plus (2.f) gives a factor

$$-\frac{1}{2}[2-(\omega-1)(\Gamma(\omega-1))^2] \tag{4.17}$$

With

$$[2-(\omega-1)(\Gamma(\omega-1))^2]\Gamma(\frac{3-\omega}{2}) = 2(5-4\gamma) \tag{4.18}$$

we find that (2.e) and (2.f) contribute to $\Pi_o^{(2)}$ with

$$-\frac{1}{96\pi^2}(\frac{4\pi}{\kappa})^2(\frac{C_2(G)}{2})^2(5-4\gamma) \tag{4.19}$$

The ghost self-energy corrections are

$$(2.g) \;=\; -\Gamma(\omega-1)(\frac{4\pi}{\kappa})^2R^{ab}\frac{1}{p^2}I_1(\omega) \tag{4.20}$$

$$(2.h) \;=\; (\frac{4\pi}{\kappa})^2R^{ab}\frac{1}{p^2}I_1(\omega) \tag{4.21}$$

$$(2.i) \;=\; -2(\frac{4\pi}{\kappa})^2(\frac{C_2(G)}{2})^2\delta^{ab}I_2(\omega) \tag{4.22}$$

$$(2.j) \;=\; (\omega-1)[\Gamma(\omega-1)]^2(\frac{4\pi}{\kappa})^2(\frac{C_2(G)}{2})^2\delta^{ab}I_2(\omega) \tag{4.23}$$

where

$$I_3(\omega) = \frac{1}{(\omega-2)(\Gamma(\omega-1))^2} I_2(\omega) \tag{4.24}$$

A similar calculation gives

$$\tilde{\Pi}^{(2)} = (\frac{4\pi}{\kappa})^2 [\frac{1}{96\pi^2}(1-\gamma)(\frac{4\pi}{\kappa})^2 R - \frac{1}{96\pi^2}(\frac{C_2(G)}{2})^2(5-4\gamma)] \tag{4.25}$$

It is remarkable that without invoking any counterterms, the singularities cancel among diagrams, leaving only finite contributions.

Unfortunately, defining the renormalization constants with (2.16), (2.25), (2.20 and (2.27), we find that either the Slavnov-Taylor identity (2.29) or the quantization condition (2.34) is *not* satisfied since $Z_A^{(2)} \neq Z_{gh}^{(2)}$ but $Z_g^{(2)} = \tilde{Z}_g^{(2)} = 1$. This means that dimensional regularization is not gauge invariant at two loops.

Using the dimensional reduction, let us consider the diagrams (2.a) – (2.k) again. We now find complete cancellation among these diagrams. For instance, in the Feynman integral expressions (2.a), (2.b) and (2.c), when the ω's in the coefficients of $I_1(\omega)$ are taken to be three, as a result of the contractions rules (2.13) and (2.14), the summation of these three diagrams gives zero. It is easy to see that similar cancellations occur between (2.e) and (2.f), (2.g) and (2.h), and (2.i) and (2.j). Consequently, the dimensional reduction gives no renormalization to the Chern-Simons action at two loops.

For the F^2 regularization at two loops, instead of performing a difficult direct calculation, we shall argue that any correction to the Chern-Simons coefficient from two (or higher) loops will break the topological quantization condition (2.34). As we have found with the F^2 regularization that the renormalized Chern-Simons coefficient at one loop is

$$\kappa_r = \kappa + \frac{1}{2} C_2(G) \mathrm{sign}(\kappa) \tag{4.26}$$

Assume that there is a correction from the next order. It will take the form

$$B\frac{1}{\kappa} \tag{4.27}$$

where B is some constant. If the quantization condition is to be satisfied, $\frac{B}{\kappa}$ must be an integer for an *arbitrary* integer κ. The only possibility is $B = 0$.

5 Renormalization of Chern-Simons-Matter Field Theory

In this Section we study perturbative renormalization of D=3 Chern-Simons gauge theory coupled to scalar and fermionic matter. We shall show that in the non-Abelian case the coefficient of the Chern-Simons term has infinite renormalization and that both the matter and the non-abelian gauge fields acquire non-vanishing anomalous dimensions at the two-loop level. However, the 2-loop β-function of the gauge coupling always vanishes, indicating that scale and conformal invariance survive quantization and infinite renormalization.

The three dimensional Euclidean actions for the coupled theories are:

$$\begin{aligned} S &= S_{SC} + S_{gf} + S_{b,f} && (5.1)\\ S_{CS} &= -i\int d^3x\ \epsilon_{\mu\nu\lambda}\left\{\frac{1}{2}A^a_\mu\partial_\nu A^a_\lambda + \frac{g_0}{6}f^{abc}A^a_\mu A^b_\nu A^c_\lambda\right\} && (5.2)\\ S_{gh} &= \int d^3x\left\{\frac{1}{2\alpha}(\partial_\mu A^a_\mu)^2 + \partial_\mu\bar{c}^a(\partial_\mu c^a + g_0 f^{abc}A^b_\mu c^c)\right\} && (5.3)\\ S_b &= \int d^3x|\partial_\mu\phi_i + g_0 T^a_{ij}A^a_\mu\phi_j|^2 && (5.4)\\ S_f &= \int d^3x\psi^\dagger_i\gamma_\mu(\partial_\mu\psi_i + g_0T^a_{ij}A^a_\mu\psi_j) && (5.5) \end{aligned}$$

Here, we have modified the Chern-Simons action (1.1) by rescaling the gauge field so that the gauge coupling constant is $g_0 = \sqrt{\frac{2\pi}{|\kappa|}}$. Gauge invariance requires that the matter fields couple with the same constant. $T^a = -(T^a)^\dagger$ are generators of the representations of the Lie algebra of G carried by the scalar ϕ or by the two-component spinor ψ [34]; the two by two antihermitean gamma matrices are defined as $\gamma_\mu = i\sigma_\mu$, $\mu = 1, 2, 3$; and a relativistic gauge $\partial_\mu A^a_\mu = 0$ is chosen with $\bar{c}$ and c the Faddeev-Popov ghosts. In (5.4) and (5.5), we have ignored the mass terms for the fermion and boson so that this model is invariant under scale and three dimensional conformal transformations.

We shall further choose the Landau gauge, with $\alpha \to 0$, where the gauge interactions are free of infrared divergence [7][9]. The Feynman rules with this gauge fixing are summarized in Fig. 3.

By power counting, it is easy to see that with the loop diagrams, the ghost and the scalar field self-energies are quadratic divergent, the gluon and the fermion field self-energies and the $\phi A \phi$ and $\bar{c}Ac$ vertices linearly divergent, while the $\phi^* A \phi A$ and the $\psi^\dagger A \psi$ vertices logarithmically divergent.

It is well known that the finite parts of loop corrections in general may depend on regularization schemes but the infinite parts do not. The infinite renormalization and the consequent results that we shall consider in this paper will be independent of the regularization used.

We introduce renormalization constants as follows

$$\begin{aligned} S &= \int \{ Z_\phi |\partial_\mu \phi|^2 + Z'_g g[(A_\mu \phi)^* \partial_\mu \phi + (\partial_\mu \phi^*) A_\mu \phi] + (Z''_g) g^2 |A_\mu \phi|^2 - \frac{i}{2} Z_A \epsilon^{\mu\nu\lambda} A^a_\mu \partial_\nu A^a_\lambda \\ &\quad - \frac{ig}{6} Z_g \epsilon^{\mu\nu\lambda} f^{abc} A^a_\mu A^b_\nu A^c_\lambda + Z_{gh} \partial_\mu \bar{c}^a \partial_\mu c^a - \tilde{Z}_g g f^{abc} \partial_\mu \bar{c}^a A^b_\mu c^c - \frac{1}{2\gamma_R} (\partial_\mu A^a_\mu)^2 \} \end{aligned} \quad (5.6)$$

where all fields are now renormalized fields. The Slavnov-Taylor identities read

$$Z''_g = (Z'_g)^2 / Z_\phi \quad (5.7)$$

$$Z_A / Z_g = Z_{gh} / \tilde{Z}_g = Z_\phi / Z'_g \quad (5.8)$$

which must be obtained in any gauge invariant regularization scheme.

In the following we shall compute the divergent parts of the renormalization constants arising from the one- and two-loop diagrams involving external or internal matter lines. For the sake of simplicity, we shall work *exclusively with regularization by dimensional reduction.* To confirm the gauge invariance of this regularization up to two-loop order, we calculate all relevant renormalization constants and verify that the Taylor-Slavnov identities (5.8) are obeyed.

For relevant group factors we shall use the following definitions:

$$tr(T^aT^b) = \delta^{ab}C_1, \quad f^{acd}f^{bcd} = \delta^{ab}C_2, \quad T^aT^a = IC_3. \tag{5.9}$$

Note that the constant C_2 is what we called $C_2(G)$ before, and only C_1 and C_3 depend on the representation R of the matter fields. All the group factors involved in two-loop calculations can be expressed in terms of these constants by the following formulas:

$$\begin{aligned}
tr(T^aT^cT^cT^b) &= C_1C_3\delta^{ab}, & (5.10)\\
tr(T^aT^cT^bT^c) &= (C_1C_3 + \frac{1}{2}C_1C_2)\delta^{ab}, & (5.11)\\
tr(T^aT^dT^c)f^{bcd} &= -\frac{1}{2}C_1C_2\delta^{ab}, & (5.12)\\
tr(T^dT^e)f^{adc}f^{bce} &= -C_1C_2\delta^{ab}. & (5.13)
\end{aligned}$$

A. One-loop Diagrams

With regularization by dimensional reduction, it is not hard to check explicitly that at one loop there is no contribution at all to any renormalization constant from the matter-gauge-field couplings.

First it is obvious that there is no one-matter-loop diagram for either the ghost self-energy or the ghost-ghost-gluon vertex.

For the one-matter-loop correction to the gluon self-energy, shown in Figs. (3.a) and (3.b), one has

$$(-ig)^2\delta^{ab}C_1\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{(2k+p)_\mu(2k+p)_\nu}{k^2(k+p)^2}. \tag{5.14}$$

It is the same in both scalar and fermion cases, since the boson-loop diagram in Fig. (3.b) vanishes identically. The integral involved

$$I_1=\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{k_\mu k_\nu}{k^2(k+p)^2} \tag{5.15}$$

with $\omega=3$ seems to be linearly divergent by power-counting. However, in dimensional regularization this integral is actually finite: Introducing Feynman parameter x in I_1 and integrating over k we have

$$I_1=\frac{1}{2}(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2})\Gamma(\frac{2-\omega}{2})/(4\pi)^{\omega/2}\Lambda^{3-\omega}\int_0^1 dx[x(1-x)p^2]^{(\omega-2)/2}. \tag{5.16}$$

This expression is finite with $\omega=3$, since $\Gamma(-1/2)=-2\pi^{1/2}$. So finally we have the one-matter-loop correction to the gluon self-energy

$$\Pi^{(1)}_{\mu\nu}(p)=\frac{g^2}{16}\delta^{ab}C_1|p|(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}) \tag{5.17}$$

For the one-matter-loop corrections (see Figs. (3.c) and (3.d)) to the three gluon vertex, the diagram involving the $\phi\phi AA$ vertex vanishes identically, and the other one (with either scalar or fermion loop) is finite, but does not contain an $\epsilon_{\mu\nu\lambda}$ factor.

As for the one-loop correction to the matter self-energy, as shown in Figs. (3.e) and (3.f), the scalar and fermion case are different. For the scalar self-energy, each diagram is separately zero, while the fermionic self-energy is finite:

$$-i\Sigma^{(1)}(p)=-i\frac{g^2}{8}p. \tag{5.18}$$

Similarly, the one-loop corrections to the matter-gluon vertex, Fig. (3.g) and (3.i) are finite, while in the scalar case the diagrams, Fig. (3.h) and (3.j), involving the $\phi\phi AA$ vertex vanish with regularization by dimensional reduction.

In summary, at one-loop order all renormalization constants Z_i can be chosen to be unity, $Z_i = 1$, at one-loop order [35] This is in agreement with ref. [7]. In particular, with our dimensional regularization there is no shift in the Chern-Simons coefficient at all and the mass squared counterterm for the scalar field $\delta m^2 = 0$, while the former is finite and the latter is linearly divergent in, e. g., F^2 regularization.

B. Divergent Parts at Two Loops with Scalar Matter

In three dimensional gauge theory, the two-loop order is the lowest order in perturbation theory at which logarithmic divergences may occur. At this order the Feynman graphs can be classified into three sets - those which vanish identically before performing any integrations, those which vanish or at least give finite results after performing one or more of the integrals (in all cases this happens solely because of Euclidean rotation invariance of the integral) and those which have divergent parts. We confine ourselves to *divergent* graphs from now on. To isolate them, certain results at one-loop order, such as eqs. (3.14), (3.15) and (3.16), are very useful. These results tell us that some one-loop diagrams, either someone alone or two of them combined together, give identically vanishing contribution before or after performing the momentum integration. When these diagrams occur as subdiagram in 2-loop diagrams, we can first perform the associated one-loop sub-integration and then show either the 2-loop diagram itself or some appropriate combination with another 2-loop diagram will give vanishing contribution. The latter is similar to the cancellation between certain pair of diagrams that we have shown in the pure Chern-Simons case (see Sec. 4). In this way one can eliminate many diagrams. For the rest, we have to carefully survey all possible diagrams and do power-counting one by one. Finally, we have been able to isolate all divergent 2-loop diagrams, as those presented in Figs. (4.a)-(4.d).

First we note that there is only one divergent 2-loop diagram for the ghost self-energy, Fig. (4.b). By using (5.18) for the one-loop gluon self-energy insertion, we can easily obtain the divergent part of this diagram and, with minimal subtraction, the ghost wave function renormalization constant

$$Z_{gh} = 1 - \frac{g^4}{48\pi^2} C_1 C_2 \frac{1}{3-\omega} \tag{5.19}$$

For the gluon wave function renormalization Z_A, there are four divergent 2-loop diagrams involving a matter loop (see Figs. (4.a-1)-(4.a-4)). To extract Z_A one needs to first contract each of them with $\epsilon_{\mu\nu\lambda} p_\lambda$. According to our rules for regularization by dimensional reduction, the contraction should be done in physical dimensions $D = 3$. Then we dimensionally continue the integrals over loop-momenta with scalar integrands. If one of the integrations actually leads to a finite result, it can be done in physical dimensions. For Fig. (4.a-1), we have

$$4\delta^{ab}(C_1 C_3 + \frac{1}{4} C_1 C_2) \frac{g^4}{p^2} \Lambda^{3-\omega} \int \frac{d^\omega k}{(2\pi)^\omega} \int \frac{d^3 q}{(2\pi)^3} \frac{k \cdot p}{k^2 q^2 (k+p+q)^2}. \tag{5.20}$$

After performing the integration over q, which leads to a finite result in $D = 3$, we are left with the following integral

$$p^2 I(\omega) \equiv \Lambda^{3-\omega} \int \frac{dk^\omega}{(2\pi)^\omega} \frac{k \cdot p}{k^2 |k+p|} \tag{5.21}$$

By introducing a Feynman parameter x, this integral can be calculated to give

$$I(\omega) = -\frac{1}{64\pi^2} \int_0^1 dx \, \sqrt{x} (\frac{1}{\omega - 3} + \ln \frac{\Lambda^2}{p^2} + \ln[x(1-x)]). \tag{5.22}$$

The term containing the pole $1/(\omega - 3)$ is the desired divergent part.

Similarly, Fig. (4.a-2) leads to

$$8\delta^{ab}(C_1 C_3 + \frac{1}{2} C_1 C_2) \frac{g^4}{p^2} \Lambda^{3-\omega} \int \frac{d^\omega k}{(2\pi)^\omega} \frac{F(k,p)}{k^2 (k+p)^2}, \tag{5.23}$$

where the function $F(k,p)$ is defined by eq. (4.7) and is evaluated in eq. (4.8). Substituting eq. (4.8) into (5.23), we find that only the second and third terms contribute to the divergent part with equal contributions, resulting in the following integral

$$-\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{2k\cdot p}{k^2|k+p|}=-p^2I(\omega). \tag{5.24}$$

with $I(\omega)$ given by eqs. (5.21) and (5.22).

Furthermore, it is easy to see that Fig. (4.a-3) leads to

$$-4\delta^{ab}C_1C_2\frac{g^4}{p^2}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{F(k,p)}{k^2(k+p)^2}, \tag{5.25}$$

which exactly cancels the C_1C_2 term in eq. (5.23).

Finally, Fig. (4.a-4) leads to, after inserting the one-loop result (5.17) for the gluon self-energy,

$$-\delta^{ab}C_1C_2\frac{1}{16}\frac{g^4}{p^2}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{p^2k^2-(k\cdot p)^2}{k^3(k+p)^2}. \tag{5.26}$$

The integral can be calculated using the Feynman parameter method which leads to a result proportional to the integral $I(\omega)$ given in eqs. (5.21) and (5.22):

$$\delta^{ab}C_1C_2\frac{g^4}{8}I(\omega). \tag{5.27}$$

Collecting all results together, we obtain the divergent part of Z_A

$$Z_A=1+\frac{g^4}{24\pi^2}C_1C_2\frac{1}{3-\omega}. \tag{5.28}$$

Note that the C_1C_3 terms are cancelled. This result is consistent with the vanishing 2-loop corrections to the divergent part of Z_A in the abelian theory [23], where one has $C_1=0$.

Without presenting more details, we list the divergent part of 2-loop corrections to the three-gluon vertex, to the self-energy of the scalar particle and to the scalar-scalar-gluon vertex diagram by diagram, respectively, in Appendix. They yield the divergent part for relevant renormalization constants as follows:

$$
\begin{aligned}
Z_g &= 1+\frac{g^4}{16\pi^2}C_1C_2\frac{1}{3-\omega} && (5.29)\\
Z_\phi &= 1+\frac{g^4}{12\pi^2}C_3(\frac{5}{2}C_3+\frac{13}{8}C_2+C_1)\frac{1}{3-\omega} && (5.30)\\
Z'_g &= 1+\frac{g^4}{12\pi^2}[C_3(\frac{5}{2}C_3+\frac{13}{8}C_2+C_1)+\frac{1}{4}C_1C_2]\frac{1}{3-\omega} && (5.31)
\end{aligned}
$$

Furthermore we have verified that $\tilde{Z}_g$ is finite, in agreement with a general theorem [33] that the ghost-ghost-gluon vertex is always finite and one can choose

$$\tilde{Z}_g = 1. \tag{5.32}$$

We have calculated these divergent parts of the renormalization constants independently without using any Slavnov-Taylor identity. But it is easy to see that they do respect the Slavnov-Taylor identities (5.8). This is a highly nontrivial check that verifies that our results obtained with regularization by dimensional reduction are compatible with gauge invariance.

Thus, the bare and renormalized fields are related by logarithmically divergent renormalization. (5.28) and (5.30) give the corresponding anomalous dimensions:

$$
\begin{aligned}
\gamma_A(g) &= -\frac{g^4}{12\pi^2}C_1C_2 && (5.33)\\
\gamma_\phi(g) &= -\frac{g^4}{6\pi^2}C_3(\frac{5}{2}C_3+\frac{13}{8}C_2+C_1) && (5.34)
\end{aligned}
$$

(These quantities, though gauge-dependent, will appear in the anomalous scale Slavnov-Taylor identities and in the calculation of anomalous dimensions for gauge invariant composite operators.) By using eqs. (5.28) and (5.29), we find that a cancellation leads to a vanishing β-function for the renormalized coupling constant $g = g_0 Z_A^{3/2}/Z_g$:

$$\beta_g(g) = 0 \tag{5.35}$$

We conjecture this is true to all orders in perturbation theory. This indicates that the physical gauge coupling constant does not run at all, though the theory needs infinite renormalization! An argument similar to that in $D = 4$ in ref. [36] shows that (5.35) also implies the conformal invariance for the Chern-Simons gauge theory coupled to massless charged matter.

In passing we note a simple criterion for checking whether the β-function vanishes or not: Since Z_i are finite in pure Chern-Simons theory, the Slavnov-Taylor identities (5.7) and (5.8) imply no infinite renormalization for g_0 if and only if the divergent contributions from matter loops satisfy

$$\delta Z_A = -2\ \delta Z_{gh} \qquad \text{or} \qquad \delta Z_g = -3\ \delta Z_{gh} \tag{5.36}$$

Since at 2 loops only one divergent diagram contributes to δZ_{gh}, to verify whether the β-function vanishes one needs to calculate only one more renormalization constant, either Z_A or Z_g.

C. Divergent Parts at Two Loops with Fermions

Similar results are obtained for the Chern-Simons theory coupled to massless fermion: We find exactly the same formulas as eqs. (5.19), (5.28) and (5.29) and (5.31) for the divergent parts of renormalization constants in the fermion case. For Z_{gh} and $\tilde{Z}_g$, this is not too surprising, since the one-loop correction to the gluon self-energy that they involve is the same for the scalar or fermion loop. But for Z_A or Z_g this needs some miraculous cancellations.

To see this, we note that there are four 2-loop diagrams for gluon self-energy which involve a fermion loop and contribute divergent parts to Z_A. See Fig (5.a-1)-(5.a-4). Comparing them with the scalar counterparts, Fig. (4.a-1)-(4.a-4), one can see that only the last diagram in each case gives the same contribution, the other three separately will not. By contracting them with $\epsilon_{\mu\nu\lambda}p_\lambda/2p^2$, we extract their contributions to the anti-symmetric part of the gluon propagator. Then by calculating the trace of Dirac matrices and first performing the integration over the loop-momentum q labeled in each diagram, we find following divergent contributions respectively for Fig. (5.a-1)-(5.a-3):

$$-\delta^{ab}C_1C_3\frac{g^4}{2p^2}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{k\cdot p+p^2}{|k|(k+p)^2}, \tag{5.37}$$

$$\delta^{ab}(C_1C_3+\frac{1}{2}C_1C_2)\frac{g^4}{8p^2}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{k\cdot p+2p^2}{|k|(k+p)^2}, \tag{5.38}$$

$$-\delta^{ab}C_1C_2\frac{g^4}{8p^2}\Lambda^{3-\omega}\int\frac{d^\omega k}{(2\pi)^\omega}\frac{2k\cdot p+p^2}{|k+p|k^2}, \tag{5.39}$$

These integrals can be easily evaluated by introducing a Feynman parameter. In Appendix, we list the results diagram by diagram for Z_A as well as for Z_ψ.

From Appendix one can see that the divergent parts of the renormalization constants Z_A, Z_{gh} and $\tilde{Z}_g$ in the case of coupling to fermions are the same as those (see eqs. (5.28), (5.19) and (5.32)) for the coupling to scalars, if they belong to the same representation R of the gauge group G. By using the first Slavnov-Taylor identity in eq. (5.8) we infer that Z_g for the fermion case should be also equal to that for the scalar case. The fermion wave function renormalization constant Z_ψ is

$$Z_\psi = 1+\frac{g^4}{16\pi^2}C_3(C_3+\frac{5}{6}C_2+\frac{1}{3}C_1)\frac{1}{3-\omega} \tag{5.40}$$

Therefore, the β-function for the gauge coupling constant g is identically zero, as in the scalar case. We did not take up the job of computing Z'_g, but exploiting the second Slavnov-Taylor identity in eq. (2.29) we can easily obtain it from eq. (5.40).

D. Implications for Abelian Chern-Simons Theory

Our results are applicable to abelian Chern-Simons theory. For the abelian theory, we take $f^{abc}=0$ and $T^a=-i$, then by minimal subtraction

$$Z_A \;=\; 1 \tag{5.41}$$

$$
\begin{aligned}
Z_\phi &= Z_g' = 1 + \frac{7g^4}{24\pi^2}\frac{1}{3-\omega} && (5.42)\\
Z_\psi &= 1 + \frac{g^4}{12\pi^2}\frac{1}{3-\omega} && (5.43)
\end{aligned}
$$

While (5.41) is known before, eqs. (5.42) and (5.43) are new results.

Since ultraviolet divergence of a gauge theory are independent of whether the coupled matter is massless or massive, our results are valid also for *massive* matter, assuming there is no bare F^2 term. In fact, in a massive theory our results for Z_i are the same as if one uses Weinberg's zero-mass renormalization scheme [37]. However, note that while massless matter does not induce a finite Maxwell or Yang-Mills term, massive matter does. If an F^2 term is incorporated in the action, the theory becomes super-renormalizable and our results can be used to derive the leading logarithmic terms in the large gauge-boson-mass limit, since the F^2 term may be viewed as a regulator for the theory without it.

6 Conclusions and Discussions

Our two-loop results have interesting and significant implications. First, eq. (5.28) show that the non-abelian gauge fields acquire an infinite renormalization from coupling to either massless or massive matter. This is in contrast to the abelian case, where the coefficient of the Chern-Simons term acquires at most finite renormalization.

The bare non-abelian coupling satisfies a topological quantization condition: $4\pi/g_0^2 =$ integer, because of the invariance of (1.1) under large gauge transformations. It is widely believed but not *a priori* clear to us [38] that the same topological quantization condition should be respected perturbatively by the renormalized (or more precisely, physical) coupling constant g. Though our vanishing β-function is consistent with a topological quantization condition for g, the latter also requires non-renormalization of g beyond 1 loop including the finite part, which is to be verified yet in Chern-Simons theory coupled to matter [39]. It is remarkable that in all cases we have examined, abelian and non-abelian, coupled to massless or massive matter, which can be either scalar or fermion, the β-function for the Chern-Simons gauge coupling always vanishes, and the 'statistics parameter' $g_0^2/4\pi$ is independent of the renormalization scale. Thus far there seems to be no unified understanding of this remarkable result: The no-renormalization theorems do not apply to the non-abelian cases, while the topological quantization does not apply to abelian cases. We speculate that the survival of scale and conformal Slavnov-Taylor identities is somehow related to the fact that the kinetic Chern-Simons action is a topological action. It is known that classical abelian Chern-Simons gauge fields coupled to quantum mechanical massive point particles are non-propagating in the sense that, at a fixed time, they can be eliminated in terms of charged currents. While the appearance of anomalous dimensions of field operators in the abelian theory would seem to imply that at the quantum level the Chern-Simons fields cease to be entirely non-dynamical when coupled to matter fields, the fact that the gauge coupling does not run indicates that the essential nondynamical role of transmuting the statistics of particles survives and is scale independent. This latter property may render these theories at least partially solvable [19].

7 Appendix: Divergent Two-Loop Contributions

In this Appendix we list our results, diagram by diagram, for the divergent part of 2-loop corrections, arising from the matter coupling, to the gluon self-energy, the three-gluon vertex, the matter self-energy and the matter-gluon vertex. First consider the bosonic case. The 2-loop divergent parts for Z_A from the four diagrams in Fig. (4.a) are respectively

$$\begin{aligned}
(4.a{-}1) &= \frac{g^4}{12\pi^2}(C_2C_3+\frac{1}{4}C_1C_2)\frac{1}{3-\omega} && (7.1)\\
(4.a{-}2) &= \frac{g^4}{12\pi^2}(C_2C_3+\frac{1}{2}C_1C_2)\frac{1}{3-\omega} && (7.2)\\
(4.a{-}3) &= \frac{g^4}{24\pi^2}C_1C_2\frac{1}{3-\omega} && (7.3)\\
(4.a{-}4) &= -\frac{g^4}{48\pi^2}C_1C_2\frac{1}{3-\omega} && (7.4)
\end{aligned}$$

Summing these contributions one obtains eq. (5.28). The four diagrams in Fig. (4.c) give rise to the following divergent parts to Z_g:

$$\begin{aligned}
(4.c{-}1) &= -\frac{g^4}{4\pi^2}(C_2C_3+\frac{1}{2}C_1C_2)\frac{1}{3-\omega} && (7.5)\\
(4.c{-}2) &= \frac{g^4}{32\pi^2}C_1C_2\frac{1}{3-\omega} && (7.6)\\
(4.c{-}3) &= \frac{g^4}{4\pi^2}(C_2C_3+\frac{3}{8}C_1C_2)\frac{1}{3-\omega} && (7.7)\\
(4.c{-}4) &= \frac{g^4}{16\pi^2}C_1C_2\frac{1}{3-\omega} && (7.8)
\end{aligned}$$

Therefore we are led to eq. (5.29).

From the four 2-loop scalar propagator diagrams in Fig. (4.d) we obtain the following divergent parts for Z_ϕ:

$$\begin{aligned}
(4.d{-}1) &= \frac{g^4}{6\pi^2}(C_3^2+\frac{1}{2}C_2C_3)\frac{1}{3-\omega} && (7.9)\\
(4.d{-}2) &= \frac{g^4}{12\pi^2}C_1C_3\frac{1}{3-\omega} && (7.10)\\
(4.d{-}3) &= \frac{g^4}{24\pi^2}(C_3^2+\frac{1}{4}C_2C_3)\frac{1}{3-\omega} && (7.11)\\
(4.d{-}4) &= \frac{g^4}{24\pi^2}C_2C_3\frac{1}{3-\omega} && (7.12)
\end{aligned}$$

These contributions are summed up to give eq. (5.30). Finally the eight diagrams in

Fig. (4.e) give the divergent contributions to Z'_g as follows:

$$(4.e{-}1) = -\frac{g^4}{6\pi^2}(C_3^2 + \frac{3}{4}C_2C_3 + \frac{1}{8}C_2^2)\frac{1}{3-\omega} \tag{7.13}$$

$$(4.e{-}2) = -\frac{g^4}{12\pi^2}(C_1C_3 + \frac{1}{4}C_1C_2)\frac{1}{3-\omega} \tag{7.14}$$

$$(4.e{-}3) = -\frac{g^4}{24\pi^2}(C_3^2 + \frac{5}{4}C_2C_3 + \frac{3}{8}C_2^2)\frac{1}{3-\omega} \tag{7.15}$$

$$(4.e{-}4) = \frac{g^4}{12\pi^2}(C_3^2 + \frac{7}{4}C_2C_3 + \frac{5}{8}C_2^2)\frac{1}{3-\omega} \tag{7.16}$$

$$(4.e{-}5) = -\frac{g^4}{12\pi^2}(C_3^2 + C_2C_3 + \frac{1}{4}C_2^2)\frac{1}{3-\omega} \tag{7.17}$$

$$(4.e{-}6) = \frac{g^4}{48\pi^2}(C_2C_3 + \frac{1}{2}C_2^2)\frac{1}{3-\omega} \tag{7.18}$$

$$(4.e{-}7) = -\frac{g^4}{24\pi^2}(C_2C_3 + \frac{1}{4}C_2^2)\frac{1}{3-\omega} \tag{7.19}$$

$$(4.e{-}8) = \frac{g^4}{192\pi^2}C_2^2\frac{1}{3-\omega} \tag{7.20}$$

These corrections give rise to eq. (5.31). We notice that the two Slavnov- Taylor identities in eq. (2.29) are respected by the above results.

For the fermionic case, the four 2-loop diagrams that give non-vanishing divergent contributions to Z_A are shown in Fig. (5.a), with

$$(5.a{-}1) = -\frac{g^4}{12\pi^2}C_1C_3\frac{1}{3-\omega} \tag{7.21}$$

$$(5.a{-}2) = \frac{g^4}{12\pi^2}(C_1C_3 + \frac{1}{2}C_1C_2)\frac{1}{3-\omega} \tag{7.22}$$

$$(5.a{-}3) = -\frac{g^4}{48\pi^2}C_1C_2\frac{1}{3-\omega} \tag{7.23}$$

$$(5.a{-}4) = -\frac{g^4}{48\pi^2}C_1C_2\frac{1}{3-\omega} \tag{7.24}$$

Their sum gives the same divergent contribution (5.28) to Z_A as the bosonic case. For Z_ψ, the four diagrams in Fig. (5.b) lead to the following divergent parts:

$$(5.b{-}1) = \frac{g^4}{48\pi^2}(C_3^2 + \frac{1}{2}C_2C_3)\frac{1}{3-\omega} \tag{7.25}$$

$$(5.b{-}2) = \frac{g^4}{24\pi^2}C_3^2\frac{1}{3-\omega} \tag{7.26}$$

$$
\begin{aligned}
(5.b\text{--}3) &= \frac{g^4}{48\pi^2} C_1 C_3 \frac{1}{3-\omega} && (7.27)\\
(5.b\text{--}4) &= \frac{g^4}{64\pi^2} C_2 C_3 \frac{1}{3-\omega} && (7.28)
\end{aligned}
$$

Summing up them we obtain the result (5.40) for Z_ψ. In the fermionic case we did not take up the job of calculating Z_g and Z'_g; rather we use the Slavnov-Taylor identities to determine them. It turns out that the 2-loop divergent part of Z_g arising from diagrams involving fermionic matter is the same as the scalar case (5.29), if the fermions and scalars belong to the same representation of the gauge group.

Figure Captions

Fig. 1. One-loop diagrams in pure Chern-Simons theory (solid line – gluon; dashed line – ghost) (a)-(c) gluon self-energy (d) ghost self-energy (e)-(g) three-gluon vertex (h)-(i) ghost-gluon vertex

Fig. 2. Two-loop diagrams in pure Chern-Simons theory (solid line – gluon; dashed line – ghost) (a)-(f) gluon self-energy (g)-(k) ghost self-energy (i)-(N) planar three-gluon vertex (o)-(T) planar ghost-gluon vertex (u)-(w) non-planar three-gluon vertex (x)-(z) non-planar ghost-gluon vertex Note that diagrams labelled by corresponding small and capital letters pairwise cancel against each other.

Fig. 3. One-loop diagrams arising from coupling to matter (wavy line – gluon, dashed line – ghost, solid line – bosonic or fermionic matter) (a)-(b) gluon self-energy (c)-(d) three gluon vertex (e)-(f) matter self-energy (g)-(j) matter-gluon vertex. Note that the diagrams (b, d, f, h, j) occur only for bosonic matter.

Fig. 4. Divergent two-loop diagrams involving bosonic matter for (a) Z_A (b) Z_gh (c) Z_g (d) Z_ϕ (e) Z_g' (wavy line – gluon, dashed line – ghost, solid line – scalar matter)

Fig. 5. Divergent two-loop diagrams involving fermionic matter for (a) Z_A (b) Z_ψ (wavy line – gluon, dashed line – ghost, solid line – fermionic matter). The diagram for Z_{gh} looks the same as Fig. 4b.

References

[1] E. Witten, Comm. Math. Phys. 121 (1989), 351.

[2] G. Moore and N.Seiberg, *Physics. Geometry and Topology*, (Plenum Publishing Corporation, 1990), - Proceedings of Banff Summer School on Particles and Fields, August 1989, 263.

[3] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B336 (1990), 581.

[4] W. Chen, G. W. Semenoff and Y.-S. Wu, *Physics, Geometry, and Topology* (Plenum Publishing Corporation, 1990) - Proceedings of Banff Summer School on Particles and Fields, August 1989, 553.

[5] W. Chen, G. W. Semenoff and Y.-S. Wu, Mod. Phys. Lett. A5 (1990), 1833.

[6] W. Chen, G. W. Semenoff and Y.-S. Wu, Phys. Rev. D44 (1991) R1625.

[7] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48(1982) 975; Ann. Phys. 140(1982) 372.

[8] W. Siegel, Nucl. Phys. B156 (1979), 135; J.J.Shonfeld, Nucl. Phys. B 185 (1981), 157; R. Jackiw and S. Templeton, Phys. Rev. D23 (1981), 2291.

[9] R. Pisarski and S. Rao, Phys. Rev. D32 (1985), 2081.

[10] W. Siegel, Phys.Lett. 84B (1979), 193; S. J. Gates, M.T.Grisaru, M. Rocek and W. Siegel, *Superspace*, Benjamin Cummings, Reading MA, 1983.

[11] L. Schulman, J. Math. Phys. 12 (1971), 304; M. Laidlaw and C.Morette-De Witt, Phys. Rev. D3 (1971), 1375; J. Leinaas and J. Myrheim, Nuovo Cinento B37 (1977), 1; G. Goldin, R. Menikoff and D. Sharp, J. Math. Phys. 22 (1981),

1664; Y.-S. Wu, Phys. Rev. Lett. 52 (1984), 2103; Y.-S. Wu, Phys. Rev. Lett. 53 (1984), 111.

[12] F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250; Y.-S. Wu and A. Zee, Phys. Lett. 147B (1984), 325; G. Semenoff, Phys. Rev. Lett. 61 (1988) 517; A. Polyakov, Mod. Phys. Lett. A3 (1988), 325; G. Semenoff and P. Sodano, Nucl. Phys. B328 (1989), 753; J. Dunne, R. Jackiw and C. Trugenberger, Ann. Phys. (N.Y.) 194 (1989), 197.

[13] A.P. Balachandran, M. Bordeau and S. Jo, Mod. Phys. Lett. A4 (1989), 1923; Int. Jour. Mod. Phys. A, 1990; J. Frohlich, *Physics, Geometry and Topology*, (Plenum Publishing Company, New York, 1990), Proceedings of the Banff Summer School on Particles and Fields, August 1989; E. Verlinde, IASSNS Preprint, 1991; G. Moore and N. Read, Nucl. Phys. B360 (1991), 362; X.G. Wen, MIT preprint, 1991, 1992.

[14] S. Girvin and R. Prange, *The Quantum Hall Effect*, Springer-Verlag, 1987; G. Semenoff and P. Sodano, Phys. Rev. Lett. 57 (1986), 1195; S Girvin and A.H. Macdonald, Phys. Rev. Lett. 57 (1987) 1252; N. Read, Phys. Rev. Lett. 62 (1989), 86; S. Zhang, T. Hansson and S. Kivelson, Phys. Rev. Let. 62 (1989), 8; D.H. Lee and S.C. Zhang, Phys. Rev. 66 (1991), 1220; Z.F. Ezawa and A. Iwazaki, Phys. Rev. B43 (1991), 2637.

[15] B. Blok and X.G. Wen, Phys. Rev. B42(1990), 8133, 8145; X.G. Wen and A. Zee, Phys. Rev. B44 (1991), 275; J. Frohlich and A. Zee, ETH (Zurich) preprint, 1991.

[16] G. Moore and N. Read, Nucl. Phys. B 360 (1991), 362.

[17] X.G. Wen, Phys. Rev Lett. 64 (1990), 2206; Phys. Rev. B43 (1991), 11025; M. Stone, Urbana preprint, 1990; Ann. Phys. (N.Y.) 207 (1991), 38; Int. J. Mod. Phys. B5 (1991), 509; A. Balatsky, Urbana preprint, 1990; Moore and Read, ref.(17); J. Frohlich and T. Kerler, Nucl. Phys. B254 (1991), 369.

[18] I. Dzyaloshingsky, A.M. Polyakov and P. Wiegman, Phys. Lett. 127A (1987) 112; V. Kalmeyer and R. Laughlin, Phys. Rev. Lett. 59 (1987), 2095; R. Laughlin, Phys. Rev. Lett. 60 (1988), 2677; A. Fetter, C. Hanna and R. Laughlin, Phys. Rev. B39 (1989), 9679; Y.H. Chen, F. Wilczek, E. Witten and B.I. Halperin, Int. J. Mod. Phys. B3 (1989), 1001; X.G. Wen and A. Zee, Phys. Rev. B41 (1990) 240; G. Semenoff and N. Weiss, Phys. Lett. 250B, 117 (1990). See references in F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific (Singapore, 1990).

[19] E.S. Fradkin and H. Palchik, Int. J. Mod. Phys. A18 (1990) 3463.

[20] A. M. Polyakov, in *Fields, Strings and Critical Phenomena*, Les Houches 1988 (North Holland); I. Dzyaloshinsky, in 1989 Summer Workshop at ICTP, Trieste (unpublished).

[21] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. 51 (1983), 2077

[22] A. N. Redlich, Phys. Rev. D29(1984), 2366.

[23] G. W. Semenoff, P. Sodano and Y.-S. Wu, Phys. Rev. Lett. 62 (1988), 715.

[24] S. Coleman and B. Hill, Phys. Lett. 159B (1985) 184; Y. Kao and M. Suzuki, Phys. Rev. D31 (1985), 2137; M. Bernstein and T. Lee, Phys. Rev. D32 (1985), 1020.

[25] W. Chen, Phys. Lett. 251B (1990), 415.

[26] V.P. Spridonov, JETP Lett. 52 (1990) No.10, 1112.

[27] V.P.Spiridonov and F.V. Tkachov, Phys. Lett. B260 (1991), 109.

[28] L. V. Avdeev, G. V. Grigoriev and D. I. Kazakov, CERN preprint TH.6091/91, 1991.

[29] Note that the following arguments for the counting of ϵ-tensors are based on naive power-counting which assumes that the regularization does not alter the structure of the propagators and vertices.

[30] The reduction of the degree of divergence by projecting out the appropriate tensor structures was called 'regularization by gauge invariant projection' in reference (8).

[31] L. Alvarez-Gaume, J. M. F. Labastida, A. V. Ramallo, Nucl.Phys. B334 (1990), 103.

[32] A. Ruiz, Proceeding of Symposium on geometrical and topological methods in field theory, Turku, Finland, 1990, edited by O.Pekonen and J.Mickelson.

[33] J. C. Taylor, Nucl. Phys. B33 (1971), 436.

[34] Even though we don't distinguish these representations here, they could be different.

[35] We therefore expect that in any gauge invariant regularization and renormalization scheme the 1-loop contribution to the constants Z_i is finite. This is known to be the case in F^2 regularization (see refs (8) and (25)).

[36] B. Schroer, Lett. Nuovo Cimento 2 (1971), 287.

[37] S. Weinberg, Phys. Rev. D8 (1973) 3497.

[38] We note that the renormalized and bare gauge transformations

$$(A_r)^U = U(x)^{-1} A_r U(x) + \frac{1}{g} U(x)^{-1} dU(x) \tag{7.29}$$

$$(A_0)^U = U(x)^{-1} A_0 U(x) + \frac{1}{g_0} U(x)^{-1} dU(x) \tag{7.30}$$

are *not* equivalent to each other in a non-abelian theory when $g \neq Z_A^{1/2} g_0$ or $Z_A \neq Z_g$. Also Slavnov-Taylor identities do not guarantee large gauge invariance for A_r.

[39] The null result by Y. C. Kao and M. Suzuki (see ref (25)) is obtained with a bare F^2 but no Chern-Simons term.